J Stat Phys (2008) 131: 117–125
DOI 10.1007/s10955-008-9496-0

Spectral Gap for Multicolor Nearest-Neighbor Exclusion Processes with Site Disorder

Azzouz Dermoune · Philippe Heinrich

Received: 17 December 2007 / Accepted: 25 January 2008 / Published online: 7 February 2008

Abstract We consider a system of multicolor disordered lattice gas in a hypercube of $\mathbb{Z}^d$. Using a recent result of Caputo (article in preparation), we give an estimate of the spectral gap for the nearest-neighbor dynamics which plays an important role in the study of hydrodynamic limit.

Keywords Spectral gap · Nearest-neighbor exclusion · Disorder

1 Introduction

Let us consider the discrete hypercube $V := \{1, \dots, L\}^d$ as sites set, with $L \geq 2$ and $d \geq 2$. The set of configurations is defined by $\Omega := \{0, 1, \dots, m\}^V$, where $m \geq 1$ is a given number of colors. A configuration of colored particles is a element $\eta = \{\eta(x),\ x \in V\}$ of Ω which can be interpreted as follows: $\eta(x) = 0$ means that the site x is empty, whereas $\eta(x) = i$ for $i \in \{1, \dots, m\}$ means that x is occupied by a particle with color i. In the sequel we will use the notation

$$\xi_x(\eta) = \mathbf{1}_{[\eta(x) \geq 1]}$$

for $x \in V$ so that $\xi(\eta) \in \{0, 1\}^V$ denotes the configuration of occupied sites associated to η. For each color i and for each configuration η, we denote by $N_i(\eta)$ the number of particles with color i on the hypercube V.

Let $\{p_x,\ x \in V\}$ be a collection of occupation probabilities. All throughout this paper, we assume that for some $\delta \in (0, \frac{1}{2}]$, we have

$$\delta \leq p_x \leq 1 - \delta \quad \text{for all } x \in V.$$

A. Dermoune (✉) · P. Heinrich
Laboratoire Paul Painlevé, UMR 8524, UFR de Mathématiques, USTL, Bât. M2,
59655 Villeneuve d'Ascq Cedex, France
e-mail: azzouz.dermoune@math.univ-lille1.fr

P. Heinrich
e-mail: philippe.heinrich@math.univ-lille1.fr

When p_x does not depend on x, we say that we are in the homogeneous case.

Let $(p_i)_{1\le i\le m}$ be a probability distribution on the color set $\{1,\dots,m\}$. Let μ be the product probability measure on Ω defined by

$$\mu(\eta)=\prod_{i=1}^{m} p_i^{N_i(\eta)}\prod_{x\in V}\left((1-p_x)^{1-\xi_x(\eta)}p_x^{\xi_x(\eta)}\right). \tag{1}$$

Take natural numbers $n_1,\dots,n_m$ such that $n_1+\dots+n_m\le L^d$. The multi-canonical measure $\nu=\nu_{n_1,\dots,n_m}$ associated to $n_1,\dots,n_m$ is obtained from μ by conditioning on the event $\{N_1=n_1,\dots,N_m=n_m\}$:

$$\nu(\cdot)=\mu\left(\cdot\mid N_1=n_1,\dots,N_m=n_m\right). \tag{2}$$

Note that ν does not depend on $(p_i)_{1\le i\le m}$. The closeness between μ and ν was studied in [2–4].

For any configuration η we write η^{xy} for the configuration where $\eta(x)$ and $\eta(y)$ have been exchanged. Let $E:=\{\{x,y\}:x,y\in V\}$ be the set of edges. For every $\gamma\ge 0$ we define a Markov generator by

$$\mathcal{L}_\gamma f(\eta)=\sum_{e\in E}c_e^\gamma(\eta)\nabla_e f(\eta), \tag{3}$$

where $\nabla_e f(\eta)=[f(\eta^e)-f(\eta)]$, and the transition rates are given by

$$c_e^\gamma(\eta)=\begin{cases} p_x(1-p_y) & \text{if } (\xi_x(\eta),\xi_y(\eta))=(0,1),\\ p_y(1-p_x) & \text{if } (\xi_x(\eta),\xi_y(\eta))=(1,0),\\ \gamma & \text{if } \xi_x(\eta)=\xi_y(\eta).\end{cases} \tag{4}$$

It is easily checked that the rates (4) make the Markov generator reversible w.r.t. μ and since the N_i's are not affected by the dynamics, the same is true for the measure ν:

$$\nu(\eta)c_e^\gamma(\eta)=\nu(\eta^e)c_e^\gamma(\eta^e)\quad \text{for all } (\eta,e)\in\Omega\times E. \tag{5}$$

If $0<n_1+\dots+n_m<L^d$ the Markov chain generated by $\mathcal{L}_\gamma$ is irreducible on the set $\{\eta:N_1(\eta)=n_1,\dots,N_m(\eta)=n_m\}$.

We shall consider the cases $\gamma=0$ and $\gamma=1$. The difference between $\mathcal{L}_0$ and $\mathcal{L}_1$ is that in $\mathcal{L}_1$ we have added the possibility of "stirring" between particles, i.e. exchange of positions of particles of different colors along an edge. Note, however, that if $m=1$ there is no difference for we have in this setting $\mathcal{L}_\gamma=\mathcal{L}_0$ for all γ.

The spectral gap, i.e. the lowest non-zero eigenvalue of $-\mathcal{L}_\gamma$, is given by

$$\lambda_\gamma(E)=\inf_{f:\mathrm{var}_\nu(f)\ne 0}\frac{\mathcal{E}_\gamma(f,f)}{\mathrm{var}_\nu(f)} \tag{6}$$

with $\mathrm{var}_\nu(f):=\nu(f^2)-\nu(f)^2$ and where $\mathcal{E}_\gamma$ is the Dirichlet form defined by

$$\mathcal{E}_\gamma(f,g)=\nu(-f\mathcal{L}_\gamma g)=\frac{1}{2}\sum_{e\in E}\nu\left[c_e^\gamma\nabla_e f\nabla_e g\right]. \tag{7}$$

It is convenient to introduce the global density:

$$\rho=\frac{n_1+\dots+n_m}{L^d}. \tag{8}$$

Now we can recall the following result due to Caputo [1]:

Theorem 1.1

(1) *Assume that* $\gamma = 1$. *There exists* $c = c(\delta) > 0$ *such that for any* L *and any* $n_1, \dots, n_m$ *with* $0 < \rho < 1$:

$$\lambda_1(E) \geq cL^d. \tag{9}$$

(2) *Assume that* $\gamma = 0$. *There exists* $c = c(\delta) > 0$ *such that for any* L *and any* $n_1, \dots, n_m$ *with* $0 < \rho < 1$:

$$\lambda_0(E) \geq c(1-\rho)L^d. \tag{10}$$

2 Our Main Results

Set $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i|$ and $\|x-y\|_0 = \max_{1\leq i\leq d} |x_i - y_i|$. Consider now two subsets of edges

$$E_1 := \{\{x,y\} \in E : \|x-y\|_1 = 1\},$$
$$E_0 := \{\{x,y\} \in E : \|x-y\|_0 = 1\},$$

and two Markov generators, called nearest-neighbor exclusion dynamics, defined as follows:

$$\mathcal{L}_1 f(\eta) = \sum_{e\in E_1} c_e^1(\eta)\nabla_e f(\eta), \tag{11}$$

$$\mathcal{L}_0 f(\eta) = \sum_{e\in E_0} c_e^0(\eta)\nabla_e f(\eta). \tag{12}$$

Let $\lambda_1(E_1)$ be the spectral gap of the dynamics defined by $\mathcal{L}_1$, and $\lambda_0(E_0)$ be the spectral gap of the dynamics defined by $\mathcal{L}_0$ with respect to the invariant measure ν defined in (2). The aim of our work is to derive an estimate of $\lambda_1(E_1)$ and $\lambda_0(E_0)$ from Caputo's results (9), (10).

Theorem 2.1 *Assume that* $\gamma = 1$. *There exists* $c(\delta, d) > 0$ *such that for all* $L, n_1, \dots, n_m$, *with* $0 < \rho < 1$,

$$\lambda_1(E_1) \geq c(\delta, d)L^{-2}. \tag{13}$$

Theorem 2.2 *Assume that* $\gamma = 0$. *Denote by* $(e_1, \dots, e_d)$ *the canonical basis of* $\mathbb{Z}^d$ *and suppose that for some integer* $T > 0$, *we have* $p_{x+Te_i} = p_x$ *for all* x *and for all* $i = 1, \dots, d$. *Then, there exists* $c(\delta, T, d) > 0$ *such that for all* $L, n_1, \dots, n_m$, *and for all* $0 < \rho \leq 1 - \frac{1}{L^d}$,

$$\lambda_0(E_0) \geq c(\delta, d, T)(1-\rho)L^{-2}. \tag{14}$$

Comments

(a) J. Quastel [5] has studied the hydrodynamic limit for the exclusion dynamics ($\gamma = 0$) along the set of edges E_1 and in the homogeneous case with two colors. An important by-product of his study is a similar result to (14). It will appear in our proof that in the non-homogeneous case the dynamics along edges E_0 makes up for the disorder, and even in the homogeneous case it is easier to handle the exchange along E_0 than the exchange along E_1.

(b) Thanks to Theorem 2.2 we can extend the hydrodynamic limit result of Wick [7] to dimension $d \geq 2$, when the occupation probabilities are T-periodic and for multicolor case.

3 Proof of Theorem 2.1

Let $x, y \in V = \{1, \ldots, L\}^d$, choose a canonical path

$$x =: x(1), x(2), \ldots, x(n(x, y)) := y,$$

with $\{x(i), x(i+1)\} \in E_1$ by moving first in the first coordinate direction, then in the second coordinate direction, and so on. The key of the proof is the following moving particle lemma similar to the one in the mono-color case [6].

Proposition 3.1 *There exists $C := C(\delta, d) > 0$ such that for all $f \in L^2(\nu)$, for all $x, y \in V$*

$$\nu(|\nabla_{xy} f|^2) \leq CL \sum_{i=1}^{n(x,y)-1} \nu(|\nabla_{x(i)x(i+1)} f|^2).$$

If Proposition 3.1 holds, then from (9)

$$\begin{aligned}
\mathrm{var}_\nu(f) &\leq CL^{-d} \sum_{\{x,y\}\subset V} \nu(|\nabla_{xy} f|^2) \\
&\leq CL^{-d+1} \sum_{\{x,y\}\subset V} \sum_{i=1}^{n(x,y)-1} \nu(|\nabla_{x(i)x(i+1)} f|^2) \\
&\leq CL^2 \sum_{e\in E_1} \nu(|\nabla_e f|^2),
\end{aligned}$$

because $n(x, y) \leq dL$ and each $\{x(i), x(i+1)\} \in E_1$ is used for at most $d(\frac{L}{2})^{d+1}$ pairs $\{x, y\}$ (see e.g. [5]). This gives Theorem 2.1.

The proof of Proposition 3.1 is based as in [6] on the following lemma.

Lemma 3.2 *Let $n \geq 3$, and $p_x = p$ for all x. We have for all $a_1, \ldots, a_n > 0$ with $\sum_{i=1}^n a_i = 1$, for all function $f \in L^2(\nu)$, and for all sequence $x(1) := x, \ldots, x(n) := y$ of elements in V,*

$$\nu(|\nabla_{xy} f|^2) \leq \sum_{i=1}^{n-1} a_i^{-1} \nu(|\nabla_{x(i)x(i+1)} f|^2).$$

Proof of Lemma 3.2. It is sufficient to show the Lemma for $n = 3$ and for $f(\eta) = f(\eta(x), \eta(y), \eta(z))$, and for the sake of simplicity we treat the case of two colors $m = 2$. We are going to show that for all $a \in (0, 1)$:

$$a(1-a)\nu(|\nabla_{xy} f|^2 \mid A_2) \leq a\nu(|\nabla_{xz} f|^2 \mid A_2) + (1-a)\nu(|\nabla_{zy} f|^2 \mid A_2) \tag{15}$$

with $A_2 = \{\eta : \{\eta(x), \eta(y), \eta(z)\} = \{r, s, s\}\}$, for $r \neq s \in \{0, 1, 2\}$, and

$$a(1-a)\nu(|\nabla_{xy} f|^2 \,|\, A_3) \le a\nu(|\nabla_{xz} f|^2 \,|\, A_3) + (1-a)\nu(|\nabla_{zy} f|^2 \,|\, A_3) \tag{16}$$

with $A_3 = \{\eta : \{\eta(x), \eta(y), \eta(z)\} = \{0, 1, 2\}\}$.

Proof of (15): Take for example $(r, s) = (1, 0)$. Set $f_1 = f(1, 0, 0)$, $f_2 = f(0, 1, 0)$, and $f_3 = f(0, 0, 1)$. We have to show that, for all $(f_1, f_2, f_3) \in \mathbb{R}^3$ and $a \in (0, 1)$, the following inequality holds:

$$a(1-a)[f_1^2 + f_3^2 - 2f_1 f_3] \le a[f_1^2 + f_2^2 - 2f_1 f_2] + (1-a)[f_2^2 + f_3^2 - 2f_2 f_3]. \tag{17}$$

The minimum on $(0, 1)$ of the function $g(a) := \frac{|f_2 - f_3|^2}{a} + \frac{|f_1 - f_2|^2}{1-a}$ is attained at $a_m = \frac{|f_2 - f_3|}{|f_2 - f_3| + |f_1 - f_2|}$, and $g(a_m) = (|f_2 - f_3| + |f_1 - f_2|)^2$. From the triangular inequality we have

$$|f_1 - f_3| \le |f_1 - f_2| + |f_2 - f_3|.$$

It follows that

$$|f_1 - f_3|^2 \le g(a_m),$$

and then (17) holds for all $a \in (0, 1)$.

Proof of (16): We set $f_1 = f(0, 1, 2)$, $f_2 = f(0, 2, 1)$, $f_3 = f(1, 0, 2)$, $f_4 = f(2, 0, 1)$, $f_5 = f(1, 2, 0)$ and $f_6 = f(2, 1, 0)$, then a simple calculus leads to the equivalent following inequality

$$\begin{aligned} a(1-a)&\left[\sum_{i=1}^{6} f_i^2 - 2f_1 f_6 - 2f_2 f_5 - 2f_3 f_4\right] \\ &\le a\left[\sum_{i=1}^{6} f_i^2 - 2f_1 f_3 - 2f_2 f_4 - 2f_5 f_6\right] \\ &\quad + (1-a)\left[\sum_{i=1}^{6} f_i^2 - 2f_1 f_2 - 2f_3 f_5 - 2f_4 f_6\right], \end{aligned}$$

for all $a \in (0, 1)$. The estimate above is equivalent to say that the matrix

$$\mathbf{M} = \begin{pmatrix} \overline{a\bar{a}} & -\bar{a} & -a & 0 & 0 & a\bar{a} \\ -\bar{a} & \overline{a\bar{a}} & 0 & -a & a\bar{a} & 0 \\ -a & 0 & \overline{a\bar{a}} & a\bar{a} & -\bar{a} & 0 \\ 0 & -a & a\bar{a} & \overline{a\bar{a}} & 0 & -\bar{a} \\ 0 & a\bar{a} & -\bar{a} & 0 & \overline{a\bar{a}} & -a \\ a\bar{a} & 0 & 0 & -\bar{a} & -a & \overline{a\bar{a}} \end{pmatrix}$$

is positive semi-definite, where $\bar{a} = 1 - a$, and

$$\overline{a\bar{a}} := 1 - a\bar{a} := 1 - a(1-a).$$

We recall the following technic of Schur complement, see for example [8], page 34. Let

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$$

be a partitioned symmetric matrix. Suppose that the matrix $A_{11} > 0$. The matrix $A \geq 0$ if and only if the Schur complement $A \setminus A_{11} \geq 0$, where

$$A \setminus A_{11} = A_{22} - A_{12}^* A_{11}^{-1} A_{12} \geq 0. \tag{18}$$

We have to show that the Schur complement of $\mathbf{M}$

$$\mathbf{M} \setminus \left[\,\overline{a\bar{a}}\,\right] \geq 0,$$

or equivalently

$$\mathbf{M^{(1)}} := \overline{a\bar{a}} A_{22} - A_{12}^* A_{12} := (m_{ij}^{(1)}) \geq 0.$$

If we set for $k \geq 1$, $\mathbf{M^{(k)}} := (m_{ij}^{(k)})$, and

$$\mathbf{M^{(k)}} = \begin{pmatrix} m_{11}^{(k)} & M_{12}^{(k)} \\ (M_{12}^{(k)})^* & M_{22}^{(k)} \end{pmatrix},$$

and we define by induction

$$\mathbf{M^{(k+1)}} := (m_{ij}^{(k+1)}) = m_{11}^{(k)} M_{22}^{(k)} - (M_{12}^{(k)})^* M_{12}^{(k)},$$

then we can show that

$$\mathbf{M^{(3)}} = 0.$$

It follows from the technic of Schur complement that $\mathbf{M^{(2)}} \geq 0$, and then $\mathbf{M^{(1)}} \geq 0$, which achieves the proof.

Remark 1 We can check that the eigenvalues of the matrix $\mathbf{M}$ are

$$2(1 - a(1-a)), 2(1 - a(1-a)), 2(1 - a(1-a)), 0, 0, 0,$$

and then $\mathbf{M} \geq 0$.

4 Proof of Theorem 2.2

In this case the analogous of the moving particle Lemma 3.2 does not work because of the exclusion. Nevertheless we will take inspiration from [5, 6]: The exchange between an empty site x and an occupied site y can be performed by a sequence of excluded nearest-neighbor exchanges along edges in E_0.

(1) *Homogeneous case*. Suppose that $p_x = p$ for all $x \in V$. First, let us explain what happens in $d = 2$ and in the following two simple cases:

Case 1: $y = x + 2e_1$ with $\eta(x) = 0$. We consider the following edges in E_0:

$$b_1 = \{x, x+e_1\}, b_2 = \{x+e_1, x+2e_1\}, b_3 = \{x+2e_1, x+e_1+e_2\},$$
$$b_4 = \{x+e_1+e_2, x\}, b_5 = \{x, x+e_1\}, b_6 = \{x+e_1, x+e_1+e_2\},$$
$$b_7 = \{x+e_1+e_2, x+2e_1\}.$$

If we set $\eta_0 = \eta$ and $\eta_i = (\eta_{i-1})^{b_i}$ for $i = 1, \ldots, 7$, then

$$\eta^{xy} = \eta_7,$$

and if we denote each edge b_i by $\{b_i^1, b_i^2\}$, we have also

$$\eta_{i-1}(b_i^1) = 0 \quad (1 \le i \le 7).$$

It follows that

$$\nu\left(|\nabla_{xy} f|^2 \mathbf{1}_{[\eta(x)=0]}\right) \le 7 \sum_{i=1}^{7} \nu\left(|\nabla_{b_i} f(\eta_{i-1})|^2 \mathbf{1}_{[\eta_{i-1}(b_i^1)=0]}\right)$$
$$\le 7 \sum_{i=1}^{7} \nu\left(|\nabla_{b_i} f|^2 \mathbf{1}_{[\eta(b_i^1)=0]}\right),$$

because ν is homogeneous.

Case 2: $y = x + 2e_1 + 2e_2$, with $\eta(x) = 0$. Let us consider the following edges in E_0,

$$b_1 = \{x, x+e_1\}, b_2 = \{x+e_1, x+2e_1+e_2\},$$
$$b_3 = \{x+2e_1+e_2, x+2e_1+2e_2\}, b_4 = \{x+2e_1+2e_2, x+e_1+e_2\},$$
$$b_5 = \{x+e_1+e_2, x+e_1\}, b_6 = \{x+e_1, x+2e_1+e_2\},$$
$$b_7 = \{x+2e_1+e_2, x+e_1+e_2\}, b_8 = \{x+e_1+e_2, x\},$$
$$b_9 = \{x, x+e_1\}, b_{10} = \{x+e_1, x+e_1+e_2\},$$
$$b_{11} = \{x+e_1+e_2, x+2e_1+2e_2\}.$$

If we set $\eta_0 = \eta$ and $\eta_i = (\eta_{i-1})^{b_i}$ for $i = 1, \ldots, 11$, then

$$\eta^{xy} = \eta_{11},$$

and if we denote each edge b_i by $\{b_i^1, b_i^2\}$, we have also

$$\eta_{i-1}(b_i^1) = 0 \quad (1 \le i \le 11).$$

It follows that

$$\nu\left(|\nabla_{xy} f|^2 \mathbf{1}_{[\eta(x)=0]}\right) \le 11 \sum_{i=1}^{11} \nu\left(|\nabla_{b_i} f(\eta_{i-1})|^2 \mathbf{1}_{[\eta_{i-1}(b_i^1)=0]}\right)$$
$$\le 11 \sum_{i=1}^{11} \nu\left(|\nabla_{b_i} f|^2 \mathbf{1}_{[\eta(b_i^1)=0]}\right),$$

because ν is homogeneous.

General case: In the general case, an analogous construction gives, for any $x, y \in V$ such that $\eta(x) = 0$, a sequence of edges $b_1, \dots, b_{n(x,y)} \in E_0$ in a "corridor" of width 1 around the canonical path joining x to y first in the first coordinate direction, then in the second coordinate direction, and so on ... and such that the sequence $(\eta_i)_{0\le i\le n(x,y)}$ defined by $\eta_0 = \eta$ and $\eta_i = (\eta_{i-1})^{b_i}$ satisfies
(a) $\eta^{xy} = \eta_{n(x,y)}, \quad \eta_{i-1}(b_i^1) = 0,$
(b) $n(x, y) \le c(d)\|x - y\|_0 \le c(d)L$ for some constant $c(d)$.
It follows that for all $x, y \in V$,

$$\nu\left(|\nabla_{xy} f|^2 \mathbf{1}_{[\eta(x)=0]}\right) \le c(d)\|x - y\|_0 \sum_{i=1}^{n(x,y)} \nu\left(|\nabla_{b_i} f|^2 \mathbf{1}_{[\eta(b_i^1)=0]}\right).$$

Denote by $\gamma(x, y)$ our path $(b_1, \dots, b_{n(x,y)})$ from x to y. We derive that

$$\begin{aligned}\sum_{x,y\in V} \nu\left(|\nabla_{xy} f|^2 \mathbf{1}_{[\eta(x)=0]}\right) &\le c(d)L \sum_{b\in E_0}\left[\sum_{x,y:b\in\gamma(x,y)}\right]\nu\left(|\nabla_b f|^2 \mathbf{1}_{[\eta(b^1)=0]}\right) \\ &\le c(d)L^{d+2} \sum_{b\in E_0} \nu\left(|\nabla_b f|^2 \mathbf{1}_{[\eta(b^1)=0]}\right),\end{aligned}$$

because each edge $b \in E_0$ is used for at most $c(d)L^{d+1}$ pairs $\{x, y\}$. From (10) we have

$$\mathrm{var}_\nu(f) \le \frac{cL^2}{1-\rho}\mathcal{E}_0(f, f).$$

(2) *T-periodic case*. Let $x, y \in V$ such that $\eta(x) = 0$. We can suppose without loss of generality that $y = x + \sum_{i=1}^d k_i T e_i$. Then $p_x = p_y = p_z$ for all z such that $\|x - z\|_0 = kT$.
Let us consider the exchange between the empty site x and the occupied site y by a sequence of excluded exchanges along edges in

$$E_T := \{b = \{b^1, b^2\} \in E : \|b^1 - b^2\|_0 = T\}.$$

We repeat word by word the construction of the homogeneous case and we get

$$\sum_{x,y\in V} \nu(|\nabla_{xy} f|^2 \mathbf{1}_{[\eta(x)=0]}) \le c(\delta, d)L^{d+2} \sum_{b\in E_T} \nu(|\nabla_b f|^2 \mathbf{1}_{[\eta(b^1)=0]}).$$

Now at a cost $c(\delta, T)$ we can suppose that the disorder is constant on any cube with width T. It follows that for each $b = \{b^1, b^2\} \in E_T$,

$$\nu(|\nabla_b f|^2 \mathbf{1}_{[\eta(b^1)=0]}) \le c(d, T) \sum_{c\in E_0(b)} \nu(|\nabla_c f|^2 \mathbf{1}_{[\eta(c^1)=0]}),$$

where $E_0(b)$ are the bonds in E_0 which belong to a "corridor" with width 1 around the canonical path from b^1 to b^2. Finally

$$\begin{aligned}\sum_{x,y\in V} \nu(|\nabla_{xy} f|^2 \mathbf{1}_{[\eta(x)=0]}) &\le c(\delta, d)c(d, T)L^{d+2} \sum_{b\in E_T}\left[\sum_{c\in E_0(b)} \nu(|\nabla_c f|^2 \mathbf{1}_{[\eta(c^1)=0]})\right] \\ &\le c(\delta, d, T)L^{d+2} \sum_{b\in E_0} \nu(|\nabla_b f|^2 \mathbf{1}_{[\eta(b^1)=0]}).\end{aligned}$$

From (10) we get

$$\mathrm{var}_\nu(f) \le \frac{cL^2}{1-\rho}\mathcal{E}_0(f, f)$$

which achieves the proof of Theorem 2.2.

Acknowledgements We thank Pietro Caputo for valuable conversations. We thank also anonymous referee for his comments which improve the presentation of our work.

References

1. Caputo, P.: On the spectral gap of the Kac walk and related evolutions (in preparation)
2. Dermoune, A., Heinrich, P.: A small step towards the hydrodynamic limit of a colored disordered lattice gas. C.R. Math. Acad. Sci. Paris, Ser. I **339**, 507–511 (2004)
3. Dermoune, A., Heinrich, P.: Equivalence of ensembles for coloured particles in a disordered lattice gas. Markov Process. Relat. Fields **11**, 405–424 (2005)
4. Dermoune, A., Martinez, S.: Around multicolour disordered lattice gas. J. Stat. Phys. **123**(1), 181–192 (2006)
5. Quastel, J.: Diffusion of color in the simple exclusion process. Commun. Pure Appl. Math. **XLV**, 623–679 (1992)
6. Quastel, J.: Bulk diffusion in a system with site disorder. Ann. Probab. **34**, 1990–2036 (2006)
7. David Wick, W.: Hydrodynamic limit of nongradient interacting particle process. J. Stat. Phys. **54**(3–4), 873–892 (1989)
8. Zhang, F.: Schur Complement and Its Applications. Springer, Berlin (2005)